

TIME REGULARITY OF THE DENSITIES FOR THE NAVIER-STOKES EQUATIONS WITH NOISE

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ABSTRACT. We prove that the density of the law of any finite dimensional projection of solutions of the Navier–Stokes equations with noise in dimension 3 is Hölder continuous in time with values in the natural space L^1 . When considered with values in Besov spaces, Hölder continuity still holds. The Hölder exponents correspond, up to arbitrarily small corrections, to the expected diffusive scaling.

1. INTRODUCTION

When dealing with a stochastic evolution PDE, the solution depends not only on the time and space independent variables, but also on the “chance” variable, that plays a completely different role. Existence of a density for the distribution of the solution is thus a form of regularity with respect to the new variable. In infinite dimension there is no canonical reference measure, therefore often existence of densities is expected for finite dimensional functionals of the solution.

This paper is a continuation of [DR14] and its aim is to give an additional understanding of the law of solutions of the Navier–Stokes equations driven by noise in dimension three. More precisely, consider the Navier–Stokes equations either on a smooth bounded domain with zero Dirichlet boundary condition or on the 3D torus with periodic boundary conditions and zero spatial mean,

$$(1.1) \quad \begin{cases} \dot{\mathbf{u}} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \dot{\eta}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases}$$

where \mathbf{u} is the velocity, p the pressure and ν the viscosity of an incompressible fluid, and $\dot{\eta}$ is Gaussian noise, white in time and coloured in space (see [Fla08] for a survey). Existence of a density for finite dimensional projections of the solution of (1.1) and its regularity in terms of Besov spaces was proved in [DR14]. In this paper we prove that those densities are almost $\frac{1}{2}$ –Hölder continuous in time with values in L^1 , as well as with values in suitable Besov spaces defined on the finite dimensional target space.

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In a way, the results we obtain in this paper are not surprising. After all we are dealing with a diffusion process and we already know from [DR14] that the density has (in terms of Besov regularity) almost one derivative. It is then expected that the time regularity is of the order of (almost) half a derivative. Likewise, if we look at the regularity of the derivative of order α , with $\alpha \in (0, 1)$, a fair expectation is that its time regularity is of order (almost) $\frac{\alpha}{2}$. On the other hand, space regularity has been obtained in a non-standard way by means of the method introduced in [DR14]. As we will see time regularity requires as well a non-trivial proof that mixes the method of [DR14] with arguments based on the Girsanov transformation. We believe that this adds value to the paper.

In a way, the problem at hand here can be considered as part of a general attempt on proving existence and regularity of densities of problems where, in principle, Malliavin calculus is not immediately applicable. Here the loss of regularity emerges due to infinite dimension. To quickly understand that Malliavin calculus is not directly applicable here, one can realize that the equation that the Malliavin derivative of the solution of (1.1) should satisfy is essentially the linearization (around 0) of (1.1). No good estimates on the linearization of (1.1) are available so far, as they could be used for uniqueness as well.

The method we use has been developed in [DR14], starting from an idea of [FP10] (see also [Rom13] for a slightly more detailed account). Later the same idea has been used in [DF13, Fou12]. An improvement of [FP10] in a different direction has been given in [BC12]. Other attempts to handle non-smooth problems are [DM11], and [KHT12, HKHY13b, HKHY13a].

2. MAIN RESULTS

2.1. Notations. If K is an Hilbert space, we denote by $\pi_F : K \rightarrow K$ the orthogonal projection of K onto a subspace $F \subset K$, and by $\text{span}[x_1, \dots, x_n]$ the subspace of K generated by its elements x_1, \dots, x_n . Given a linear operator $\mathcal{Q} : K \rightarrow K'$, we denote by \mathcal{Q}^* its adjoint.

2.1.1. Function spaces. We recall the definition of Besov spaces. The general definition is based on the Littlewood–Paley decomposition, but it is not the best suited for our purposes. We shall use an alternative equivalent definition (see [Tri83, Tri92]) in terms of differences. Given $f : \mathbf{R}^d \rightarrow \mathbf{R}$, define

$$\begin{aligned} (\Delta_h^1 f)(x) &= f(x+h) - f(x), \\ (\Delta_h^n f)(x) &= \Delta_h^1 (\Delta_h^{n-1} f)(x) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} f(x+jh), \end{aligned}$$

and, for $s > 0$, $1 \leq p \leq \infty$, $1 \leq q < \infty$,

$$[f]_{B_{p,q}^s} = \left(\int_{\{|h| \leq 1\}} \frac{\|\Delta_h^n f\|_{L^p}^q}{|h|^{sq}} \frac{dh}{|h|^d} \right)^{\frac{1}{q}},$$

and for $q = \infty$,

$$[f]_{B_{p,\infty}^s} = \sup_{|h| \leq 1} \frac{\|\Delta_h^n f\|_{L^p}}{|h|^s},$$

where n is any integer larger than s . Given $s > 0$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, define

$$B_{p,q}^s(\mathbf{R}^d) = \{f : \|f\|_{L^p} + [f]_{B_{p,q}^s} < \infty\}.$$

This is a Banach space when endowed with the norm $\|f\|_{B_{p,q}^s} := \|f\|_{L^p} + [f]_{B_{p,q}^s}$.

When in particular $p = q = \infty$ and $s \in (0, 1)$, the Besov space $B_{\infty,\infty}^s(\mathbf{R}^d)$ coincides with the Hölder space $C_b^s(\mathbf{R}^d)$, and in that case we will denote by $\|\cdot\|_{C_b^s}$ and $[\cdot]_{C_b^s}$ the corresponding norm and semi-norm.

2.1.2. Navier Stokes framework. Let H be the standard space of square summable divergence free vector fields, defined as the closure of divergence free smooth vector fields satisfying the boundary condition (either zero Dirichlet or periodic, with zero spatial mean in the latter case), with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$. Define likewise V as the closure of the same space of test functions with respect to the H^1 norm.

Let Π_L be the Leray projector, $A = -\Pi_L \Delta$ the Stokes operator, and denote by $(\lambda_k)_{k \geq 1}$ and $(e_k)_{k \geq 1}$ the eigenvalues and the corresponding orthonormal basis of eigenvectors of A . Define the bi-linear operator $B : V \times V \rightarrow V'$ as $B(u, v) = \Pi_L(u \cdot \nabla v)$, $u, v \in V$, and recall that $\langle u_1, B(u_2, u_3) \rangle = -\langle u_3, B(u_2, u_1) \rangle$. We refer to Temam [Tem95] for a detailed account of all the above definitions.

The noise $\dot{\eta} = \mathcal{S}W$ in (1.1) is coloured in space by a covariance operator $\mathcal{S}^* \mathcal{S} \in \mathcal{L}(H)$, where W is a cylindrical Wiener process (see [DPZ92] for further details). We assume that $\mathcal{S}^* \mathcal{S}$ is trace-class and we denote by $\sigma^2 = \text{Tr}(\mathcal{S}^* \mathcal{S})$ its trace. Finally, consider the sequence $(\sigma_k^2)_{k \geq 1}$ of eigenvalues of $\mathcal{S}^* \mathcal{S}$, and let $(q_k)_{k \geq 1}$ be the orthonormal basis in H of eigenvectors of $\mathcal{S}^* \mathcal{S}$.

2.2. Galerkin approximations. With the above notations, we can recast problem (1.1) as an abstract stochastic equation,

$$(2.1) \quad du + (vAu + B(u)) dt = \mathcal{S} dW,$$

with initial condition $u(0) = x \in H$. It is well-known [Fla08] that for every $x \in H$ there exist a martingale solution of this equation, that is a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0})$, a cylindrical Wiener process \tilde{W} and a process u with trajectories in $C([0, \infty); D(A)') \cap L_{\text{loc}}^\infty([0, \infty), H) \cap L_{\text{loc}}^2([0, \infty); V)$ adapted to $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ such that the above equation is satisfied with \tilde{W} replacing W .

We will consider in particular solutions of (1.1) obtained as limits of Galerkin approximations. Given an integer $N \geq 1$, denote by H_N the sub-space $H_N = \text{span}[e_1, \dots, e_N]$ and denote by $\pi_N = \pi_{H_N}$ the projection onto H_N . It is standard (see for instance [Fla08]) to verify that the problem

$$(2.2) \quad du^N + (vAu^N + B^N(u^N)) dt = \pi_N \mathcal{S} dW,$$

where $B^N(\cdot) = \pi_N B(\pi_N \cdot)$, admits a unique strong solution for every initial condition $x^N \in H_N$. Moreover,

$$(2.3) \quad \mathbb{E} \left[\sup_{[0, T]} \|u^N\|_H^p \right] \leq c_p (1 + \|x^N\|_H^p),$$

for every $p \geq 1$ and $T > 0$, where c_p depends only on p , T and the trace of $\mathcal{S}\mathcal{S}^*$.

If $x \in H$, $x^N = \pi_N x$ and \mathbb{P}_x^N is the distribution of the solution of the problem above with initial condition x^N , then any limit point of $(\mathbb{P}_x^N)_{N \geq 1}$ is a solution of the martingale problem associated to (1.1) with initial condition x .

Remark 2.1. In general, there is nothing special with the basis provided by the eigenvectors of the Stokes operator and our results would work when applied to Galerkin approximations generated by any (smooth enough) orthonormal basis of H . The crucial assumption is that the solution is a limit point of finite dimensional approximations. Some of the results concerning densities (but not those in this paper) can be generalized to any martingale weak solution of (2.1), see [Rom14].

2.3. Assumptions on the covariance. Given a finite dimensional subspace F of H , we assume the following non degeneracy condition on the covariance,

$$(2.4) \quad \mathcal{S}x = f \quad \text{has a solution for every } f \in F,$$

The condition above is stronger than the condition

$$(2.5) \quad \pi_F \mathcal{S} \mathcal{S}^* \pi_F \quad \text{is a non-singular matrix,}$$

used in [DR14] to prove bounds on the Besov norm of the density. It is not clear if our results here may be true under the weaker assumption (2.5) (see Remark 4.6 though).

Indeed, for our method — that works through finite dimensional approximations, it is convenient to assume a slightly stronger version of (2.4), namely that

$$(2.6) \quad \pi_N \mathcal{S}x = f \quad \text{has a solution for every } f \in F,$$

for N large enough.

2.4. Continuity in time of the density. Our first main result is that densities of finite dimensional projections of solutions of (2.1) are continuous (actually Hölder with exponent almost $\frac{1}{2}$) with respect to time with values in the natural space L^1 of densities.

Theorem 2.2. *Let F be a finite dimensional subspace of $D(A)$ generated by a finite set of eigenvalues of the Stokes operator, and assume (2.6).*

Given $\alpha \in (0, 1)$, there is $c_1 > 0$ such that if $x \in H$ and u is a weak solution of (2.1) with initial condition x that is a limit point of Galerkin approximations, if $f(\cdot; x)$ is the

density with respect to the Lebesgue measure on F of the random variable $\pi_F u(\cdot)$, then

$$\|f(t; x) - f(s; x)\|_{L^1(F)} \leq c_1 (1 + s \vee t)^{\frac{1-\alpha}{2}} \|f(s \wedge t)\|_{B_{1,\infty}^\alpha} (1 + \|x\|_H^2)^2 |t - s|^{\frac{\alpha}{2}},$$

for every $s, t > 0$.

The theorem above follows immediately from Proposition 3.1 and lower semi-continuity. Notice that the term $\|f(s \wedge t)\|_{B_{1,\infty}^\alpha}$ is singular when $s \wedge t$ approaches 0 (see Lemma 4.3).

By trading time-continuity with space-time continuity, we can obtain an estimate similar to the one given in the above theorem for the Besov norm of the density.

Theorem 2.3. *Let F be a finite dimensional subspace of $D(A)$ generated by a finite set of eigenvalues of the Stokes operator, and assume (2.6).*

Given $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, there is $c_2 > 0$ such that if $x \in H$ and u is a weak solution of (2.1) with initial condition x that is a limit point of Galerkin approximations, if $f(\cdot; x)$ is the density with respect to the Lebesgue measure on F of the random variable $\pi_F u(\cdot)$, then

$$\|f(t; x) - f(s; x)\|_{B_{1,\infty}^\alpha} \leq c_2 |t - s|^{\frac{\beta}{2}},$$

for every $s, t > 0$, where

$$c_2 \approx (1 + s \vee t)^{\frac{1-\beta}{2}} (1 + \|x\|_H^2)^3 ([f(t)]_{B_{1,\infty}^{1-\delta}} + [f(s)]_{B_{1,\infty}^{1-\delta}}),$$

and $\delta < 1 - (\alpha + \beta)$.

The proof of this theorem is given by means of Proposition 4.2. A crucial tool in the proof of both theorems is Girsanov's transformation. This explains why we need the slightly stronger assumption (2.4) rather than the assumption (2.5) used in [DR14]. Girsanov's change of measure is used to perform a sort of fractional integration by parts and move the tiny regularity from space to time (see Lemma 3.6).

3. THE ESTIMATE IN L^1

This section is devoted to the proof of the Hölder estimate of the density with values in L^1 . A classical way is to derive first some space regularity and then use it to prove the time regularity. In a way, this is also the bulk of our method, although due to the low regularity we have at hand (see Lemma 4.3), this can be done only after a suitable simplification. The main tool we use here is the Girsanov transformation and the logarithmic moments of the Girsanov density. The version of the Girsanov theorem we use follows from [LS01, Chapter 7]. The main result of this section is as follows.

Proposition 3.1. *Let F be a finite dimensional subspace of $D(A)$ generated by a finite set of eigenvalues of the Stokes operator, and assume (2.6). Given $\alpha \in (0, 1)$, there is $c_3 > 0$ such that if $x \in H$, N is large enough (that $F \subset H_N$) and u^N is a solution of (2.2) with initial condition $\pi_N x$, if $f_N(\cdot; x)$ is the density with respect to the Lebesgue measure on F of $\pi_F u^N(\cdot)$, then*

$$\|f_N(t) - f_N(s)\|_{L^1(F)} \leq c_3(1 + s \vee t)^{\frac{1-\alpha}{2}} \|f_N(s \wedge t)\|_{B_{1,\infty}^\alpha} (1 + \|x\|_H^2)^2 |t - s|^{\frac{\alpha}{2}},$$

for every $s, t > 0$.

In the rest of the section we will drop, for simplicity and to make the notation less cumbersome, the index N . It is granted though that we work with solutions of the Galerkin system (2.2).

3.1. The Girsanov equivalence. Let us assume now (2.6) and consider the following two stochastic equations on H_N

$$\begin{aligned} du + (vAu + \pi_N B(u)) dt &= \pi_N S dW, \\ dv + (\pi_N - \pi_F)(vAv + B(v)) dt &= \pi_N S dW. \end{aligned}$$

It is easy to see that both equations have a unique strong solution for every initial condition in H_N . In view of the application of the Girsanov transformation, assume $u(0) = v(0) \in H_N$.

3.1.1. The Moore–Penrose pseudo–inverse. Given a linear bounded operator $S : H \rightarrow H$ and a finite dimensional subspace $F \subset H$ such that $Sx = f$ has at least one solution for every $f \in F$, define

$$S^+ f = \arg \min\{\|x\|_H : x \in H \text{ and } Sx = f\}.$$

It is elementary to check that the pseudo–inverse $S^+ : F \rightarrow H$ is well defined and is a linear bounded operator, since given f the minima x are characterized by $\langle x, y - x \rangle_H \geq 0$ for every $y \in H$ such that $Sy = f$. In particular $SS^+ f = f$ and, if Assumption (2.6) holds for S , $(\pi_N S)^+ = S^+$.

3.1.2. Reduction by the Girsanov transformation. Fix for the rest of the section $T > 0$. If $w \in C([0, T]; H_N)$, set

$$\tau_n(w) = \inf\left\{t \leq T : \int_0^t \|S^+ \pi_F(vAw + B(w))\|_H^2 ds \geq n\right\},$$

and $\tau_n(w) = T$ if the above set is empty, and $\chi_t^n(w) = \mathbb{1}_{\{\tau_n(w) \geq t\}}$. By (2.3) $\tau_n(u) < \infty$ almost surely. Similar computations yield that also $\tau_n(v) < \infty$ almost surely.

Let v^n be the solution of

$$\begin{aligned} v^n(t) = v(t \wedge \tau_n(v)) - \int_0^t (1 - \chi_s^n(v)) \pi_N(vAv^n + B(v^n)) ds + \\ + \int_0^t (1 - \chi_s^n(v)) \pi_N \mathcal{S} dW, \end{aligned}$$

then $v^n(t) = v(t)$ on $\{\tau_n(v) \geq t\}$, $\tau_t^n(v) = \tau_t^n(v^n)$, and $v^n(t) \rightarrow v(t)$ almost surely. More precisely, $v^n(t) = v(t)$ for n large enough (ω -wise), therefore $\phi(v^n(t)) \rightarrow \phi(v(t))$ almost surely for any bounded measurable ϕ .

Moreover, since

$$v(t \wedge \tau_n(v)) = v(0) - \int_0^t \chi_s^n(v) (\pi_N - \pi_F)(vAv + B(v)) ds + \int_0^t \chi_s^n(v) \pi_N \mathcal{S} dW,$$

it follows that

$$\begin{aligned} v^n(t) = v(0) - \int_0^t (vAv^n + \pi_N B(v^n)) ds + \\ + \int_0^t \pi_N \mathcal{S} dW + \int_0^t \chi_s^n(v^n) \pi_F(vAv^n + B(v^n)) ds. \end{aligned}$$

By the Girsanov theorem the process

$$\begin{aligned} G_t^n = \exp \left(\int_0^t \chi_s^n(v^n) \mathcal{S}^+ \pi_F(vAv^n + B(v^n)) dW_s + \right. \\ \left. - \frac{1}{2} \int_0^t \chi_s^n(v^n) \|\mathcal{S}^+ \pi_F(vAv^n + B(v^n))\|_H^2 ds \right) \end{aligned}$$

is a martingale and the law of u on $[0, T]$ with respect to the original probability measure \mathbb{P} is equal to the law of v^n on $[0, T]$ with respect to the new probability measure $G_T^n \mathbb{P}$.

3.2. Increments of the Girsanov density. In this section we estimate the time increments of the Girsanov density. This provides half of the proof of Proposition 3.1.

Lemma 3.2. *There is $c_4 > 0$ such that for every $0 \leq s \leq t \leq T$ and every $n \geq 1$,*

$$\mathbb{E} \left[G_t^n \left| \log \frac{G_t^n}{G_s^n} \right| \right] \leq c_4 (t - s)^{\frac{1}{2}} (1 + \|u(0)\|_H^2)^2.$$

Proof. By changing back the probability measure, since on the interval $[0, t]$ u under \mathbb{P} has the same law of v^n under $G_t^n \mathbb{P}$,

$$\begin{aligned} \mathbb{E} \left[G_t^n \left| \log \frac{G_t^n}{G_s^n} \right| \right] &= \mathbb{E} \left[\left| \log \frac{G_t^n(u)}{G_s^n(u)} \right| \right] \\ &\leq \mathbb{E} \left[2 \left| \int_s^t \chi_r^n(u) \mathcal{S}^+ \pi_F(vAu + B(u)) \, dW_r \right| \right] \\ &\quad + \mathbb{E} \left[\int_s^t \chi_r^n(u) \|\mathcal{S}^+ \pi_F(vAu + B(u))\|_H^2 \, dr \right] \\ &\leq c_4 (t-s)^{\frac{1}{2}} (1 + \|u(0)\|_H^2)^2, \end{aligned}$$

where we have used the Burkholder-Davis-Gundy inequality and (2.3). \square

Lemma 3.3. *There is $c_5 > 0$ such that for every $0 \leq s \leq t \leq T$ and $n \geq 1$,*

$$|\mathbb{E}[(G_t^n - G_s^n)X]| \leq c_5 \|X\|_\infty (1 + \|u(0)\|_H^2)^2 (t-s)^{\frac{1}{2}},$$

where X is any real bounded random variable.

Proof. Without loss of generality, we assume $\|X\|_\infty \leq 1$. Fix $0 \leq s \leq t \leq T$ and notice that, since G_t^n is a martingale, $\mathbb{E}[G_t^n - G_s^n] = 0$, hence

$$\mathbb{E}[(G_t^n - G_s^n) \mathbb{1}_{\{G_t^n \geq G_s^n\}}] = \mathbb{E}[(G_s^n - G_t^n) \mathbb{1}_{\{G_s^n \geq G_t^n\}}].$$

Thus

$$\begin{aligned} |\mathbb{E}[(G_t^n - G_s^n)X]| &= |\mathbb{E}[(G_t^n - G_s^n)X \mathbb{1}_{\{G_t^n \geq G_s^n\}}] + \mathbb{E}[(G_t^n - G_s^n)X \mathbb{1}_{\{G_s^n \geq G_t^n\}}]| \\ &\leq \mathbb{E}[(G_t^n - G_s^n) \mathbb{1}_{\{G_t^n \geq G_s^n\}}] + \mathbb{E}[(G_s^n - G_t^n) \mathbb{1}_{\{G_s^n \geq G_t^n\}}] \\ &= 2\mathbb{E}[(G_t^n - G_s^n) \mathbb{1}_{\{G_t^n \geq G_s^n\}}] \\ &= 2\mathbb{E} \left[G_s \left(e^{\log \frac{G_t^n}{G_s^n}} - 1 \right) \mathbb{1}_{\{G_t^n \geq G_s^n\}} \right], \end{aligned}$$

and, by using the elementary inequality $e^x - 1 \leq (1 \wedge |x|) e^x$, $x \in \mathbf{R}$,

$$\begin{aligned} |\mathbb{E}[(G_t^n - G_s^n)X]| &\leq 2\mathbb{E} \left[G_s \left(e^{\log \frac{G_t^n}{G_s^n}} - 1 \right) \mathbb{1}_{\{G_t^n \geq G_s^n\}} \right] \\ &\leq 2\mathbb{E} \left[G_s \left(1 \wedge \log \frac{G_t^n}{G_s^n} \right) \frac{G_t^n}{G_s^n} \mathbb{1}_{\{G_t^n \geq G_s^n\}} \right] \\ &\leq 2\mathbb{E} \left[G_t^n \left(1 \wedge \left| \log \frac{G_t^n}{G_s^n} \right| \right) \right] \\ &\leq 2\mathbb{E} \left[G_t^n \left| \log \frac{G_t^n}{G_s^n} \right| \right]. \end{aligned}$$

Finally, the conclusion of the lemma follows by Lemma 3.2. \square

3.3. Proof of Proposition 3.1. We recall an elementary inequality, its proof is straightforward calculus: for every $x, y \geq 0$ and $\epsilon > 0$,

$$(3.1) \quad xy \leq \epsilon e^{\frac{y}{\epsilon}} + \epsilon x \log x.$$

Lemma 3.4. *For every $\epsilon > 0$, every $s, t \in [0, T]$, every $n \geq 1$ and every bounded measurable $\phi : F \rightarrow \mathbf{R}$,*

$$|\mathbb{E}[G_s^n (\phi(\pi_F v^n(t)) - \phi(\pi_F v(t)))]| \leq \epsilon \|\phi\|_\infty (c_4 \sqrt{T} (1 + \|u(0)\|_H^2)^2 + e^{\frac{2}{\epsilon}} \mathbb{P}[\tau_n(v) < t]).$$

Proof. Fix $\epsilon > 0$ and assume for simplicity $\|\phi\|_\infty \leq 1$. We know that $v^n(t) = v(t)$ on $\tau_n(v) \geq t$, hence

$$\mathbb{E}[G_s^n (\phi(\pi_F v^n(t)) - \phi(\pi_F v(t)))] = \mathbb{E}[G_s^n (\phi(\pi_F v^n(t)) - \phi(\pi_F v(t))) \mathbb{1}_{\{\tau_n(v) < t\}}].$$

By the inequality (3.1) above, applied to $x = G_s^n$ and $y = \frac{1}{\epsilon} (\phi(\pi_F v^n(t)) - \phi(\pi_F v(t)))$,

$$\begin{aligned} \mathbb{E}[G_s^n (\phi(\pi_F v^n(t)) - \phi(\pi_F v(t))) \mathbb{1}_{\{\tau_n(v) < t\}}] &\leq \\ &\leq \epsilon \mathbb{E}[G_s^n \log G_s^n] + \epsilon \mathbb{E}[e^{\phi(\pi_F v^n(t)) - \phi(\pi_F v(t))} \mathbb{1}_{\{\tau_n(v) < t\}}] \leq \\ &\leq \epsilon \mathbb{E}[G_s^n \log G_s^n] + \epsilon e^{\frac{2}{\epsilon}} \mathbb{P}[\tau_n(v) < t]. \end{aligned}$$

The statement of the lemma now follows by Lemma 3.2. \square

Let U_ϕ be the solution of the heat equation

$$(3.2) \quad \partial_t U_\phi = \frac{1}{2} \text{Tr}(\pi_F \mathcal{S} (\pi_F \mathcal{S})^* D^2 U_\phi),$$

with initial condition ϕ . This is well defined, smooth and a linear transformation of the standard heat equation due again to assumption (2.5).

Lemma 3.5. *For every $0 \leq s \leq t \leq T$, $n \geq 1$ and $\phi : F \rightarrow \mathbf{R}$ bounded measurable,*

$$\mathbb{E}[G_s^n \phi(\pi_F v(t))] = \mathbb{E}[G_s^n U_\phi(t - s, \pi_F v(s))].$$

Proof. Set $\beta(t) = \pi_F v(t)$, then by assumption (2.5) $\beta(t) = \pi_F u(0) + \int_0^t \pi_F \mathcal{S} dW$ is a d -dimensional Brownian motion started at $\pi_F u(0)$. By the Markov property,

$$\mathbb{E}[G_s^n \phi(\pi_F v(t))] = \mathbb{E}[G_s^n \mathbb{E}[\phi(\beta(t)) | \mathcal{F}_s]] = \mathbb{E}[G_s^n U_\phi(t - s, \beta_s)]. \quad \square$$

Lemma 3.6. *There is $c_6 > 0$ such that for every $0 \leq s \leq t \leq T$, every $n \geq 1$, every bounded measurable $\phi : F \rightarrow \mathbf{R}$, and every $\alpha \in (0, 1)$,*

$$\begin{aligned} \mathbb{E}[G_s^n (\phi(\pi_F v^n(t)) - \phi(\pi_F v^n(s)))] &\leq c_6 \|\phi\|_\infty ([f(s)]_{B_{1,\infty}^\alpha}(t - s)^{\frac{\alpha}{2}} \\ &\quad + \epsilon \sqrt{T} (1 + \|u(0)\|_H^2)^2 + \epsilon e^{\frac{2}{\epsilon}} \mathbb{P}[\tau_n(v) < t]). \end{aligned}$$

Proof. Let s, t, n, ϕ as in the statement of the lemma and assume for simplicity $\|\phi\|_\infty \leq 1$. We have

$$\begin{aligned} \mathbb{E}[G_s^n(\phi(\pi_F v^n(t)) - \phi(\pi_F v^n(s)))] &= \underbrace{\mathbb{E}[G_s^n(\phi(\pi_F v^n(t)) - U_\phi(t-s, \pi_F v^n(s)))]}_{\boxed{a}} \\ &\quad + \underbrace{\mathbb{E}[G_s^n(U_\phi(t-s, \pi_F v^n(s)) - \phi(\pi_F v^n(s)))]}_{\boxed{b}}. \end{aligned}$$

For the first term we use Lemma 3.5, Lemma 3.4 twice, and $\|U_\phi\|_\infty \leq \|\phi\|_\infty$,

$$\begin{aligned} \boxed{a} &= \mathbb{E}[G_s^n(\phi(\pi_F v^n(t)) - \phi(\pi_F v(t)))] + \mathbb{E}[G_s^n(\phi(\pi_F v(t)) - U_\phi(t-s, \pi_F v(s)))] \\ &\quad + \mathbb{E}[G_s^n(U_\phi(t-s, \pi_F v(s)) - U_\phi(t-s, \pi_F v^n(s)))] \\ &\leq 2\epsilon(c_4\sqrt{T}(1 + \|u(0)\|_H^2) + e^{\frac{2}{\epsilon}} \mathbb{P}[\tau_n(v) < t]). \end{aligned}$$

For the second term, we change back the probability measure, since on the interval $[0, s]$ u under \mathbb{P} has the same law of v^n under $G_s^n\mathbb{P}$,

$$\begin{aligned} \boxed{b} &= \mathbb{E}[(U_\phi(t-s, \pi_F u(s)) - \phi(\pi_F u(s)))] \\ &= \int_{\mathbf{R}^d} (U_\phi(t-s, y) - \phi(y)) f(s, y) \, dy \\ &= \int_{\mathbf{R}^d} (\hat{\mathbb{E}}[\phi(y + \hat{B}_{t-s})] - \phi(y)) f(s, y) \, dy \\ &= \hat{\mathbb{E}} \left[\int_{\mathbf{R}^d} \phi(y) (f(s, y - \hat{B}_{t-s}) - f(s, y)) \, dy \right] \\ &\leq \hat{\mathbb{E}}[\|f(s, \cdot - \hat{B}_{t-s}) - f(s, \cdot)\|_{L^1(\mathbf{R}^d)}] \\ &\leq [f(s)]_{B_{1,\infty}^\alpha} \hat{\mathbb{E}}[|\hat{B}_{t-s}|^\alpha] \\ &\leq c_7 [f(s)]_{B_{1,\infty}^\alpha} (t-s)^{\frac{\alpha}{2}}, \end{aligned}$$

where $\alpha \in (0, 1)$, $f(t, \cdot)$ (or more precisely $f_N(t, \cdot)$, but again we drop the superscript for simplicity) is the density of $\pi_F u(t)$, and where $(\hat{B}_t)_{t \geq 0}$ is an independent F -valued Brownian motion with (spatial) covariance $\pi_F \mathcal{S}(\pi_F \mathcal{S})^*$ introduced to represent the solutions of (3.2). \square

We finally have all the ingredients to complete the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $0 \leq s \leq t$. By duality, it is sufficient to estimate the following quantity for each bounded measurable $\phi : F \rightarrow \mathbf{R}$ with $\|\phi\|_\infty \leq 1$. For

every $n \geq 1$, by the Girsanov transformation detailed in Section 3.1,

$$\begin{aligned}
\int_{\mathbb{F}} \phi(y)(f(t, y) - f(s, y)) \, dy &= \mathbb{E}[\phi(\pi_{\mathbb{F}} u(t)) - \phi(\pi_{\mathbb{F}} u(s))] \\
&= \mathbb{E}[G_t^n(\phi(\pi_{\mathbb{F}} v^n(t)) - \phi(\pi_{\mathbb{F}} v^n(s)))] \\
&= \mathbb{E}[G_t^n \phi(\pi_{\mathbb{F}} v^n(t)) - G_s^n \phi(\pi_{\mathbb{F}} v^n(s))] \\
&= \underbrace{\mathbb{E}[(G_t^n - G_s^n) \phi(\pi_{\mathbb{F}} v^n(t))]}_{\boxed{1}} \\
&\quad + \underbrace{\mathbb{E}[G_s^n(\phi(\pi_{\mathbb{F}} v^n(t)) - \phi(\pi_{\mathbb{F}} v^n(s)))]}_{\boxed{2}}.
\end{aligned}$$

The first term is estimated through Lemma 3.3,

$$\boxed{1} \leq c_5(1 + \|x\|_H^2)^2(t - s)^{\frac{1}{2}},$$

the second term through Lemma 3.6, for every $\epsilon > 0$,

$$\boxed{2} \leq c_6([f(s)]_{B_{1,\infty}^\alpha}(t - s)^{\frac{\alpha}{2}} + \epsilon \sqrt{t}(1 + \|x\|_H^2)^2 + \epsilon e^{\frac{2}{\epsilon}} \mathbb{P}[\tau_n(v) < t]),$$

so that in conclusion

$$\begin{aligned}
\left| \int_{\mathbb{F}} \phi(y)(f(t, y) - f(s, y)) \, dy \right| &\leq c_5(1 + \|x\|_H^2)^2(t - s)^{\frac{1}{2}} + \\
&\quad + c_6([f(s)]_{B_{1,\infty}^\alpha}(t - s)^{\frac{\alpha}{2}} + \epsilon \sqrt{t}(1 + \|x\|_H^2)^2 + \epsilon e^{\frac{2}{\epsilon}} \mathbb{P}[\tau_n(v) < t]),
\end{aligned}$$

and by taking first the limit as $n \uparrow \infty$, so that $\mathbb{P}[\tau_n(v) < t] \downarrow 0$, and then as $\epsilon \downarrow 0$, the statement of the proposition follows. \square

4. THE ESTIMATE IN THE BESOV SEMINORM

In this section we prove Theorem 2.3. To this end we use together the machinery on Girsanov's theorem introduced in the previous section and the technique based on Besov spaces introduced in [DR14].

4.1. A smoothing lemma. The technique introduced in [DR14] is based on a duality estimate that provides a quantitative integration by parts. Since we are dealing with regularity properties of low order, we will use Besov spaces to measure it. The following lemma is implicitly given in [DR14], we state it here explicitly and give a complete proof.

Lemma 4.1 (smoothing lemma). *If μ is a finite measure on \mathbb{R}^d and there are an integer $m \geq 1$, two real numbers $s > 0$, $\gamma \in (0, 1)$, with $\gamma < s < m$, and a constant $K > 0$ such that for every $\phi \in C_b^\gamma(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$,*

$$\left| \int_{\mathbb{R}^d} \Delta_h^m \phi(x) \mu(dx) \right| \leq K|h|^s \|\phi\|_{C_b^\gamma},$$

then μ has a density f_μ with respect to the Lebesgue measure on \mathbf{R}^d . Moreover, for every $r < s - \gamma$ there exists $c_8 > 0$ such that

$$(4.1) \quad \|f_\mu\|_{B_{1,\infty}^\gamma} \leq c_8(\mu(\mathbf{R}^d) + K).$$

Proof. Fix a smooth function ϕ . Let $(\varphi_\epsilon)_{\epsilon>0}$ be a smoothing kernel, namely $\varphi_\epsilon = \epsilon^{-d}\varphi(x/\epsilon)$, with $\varphi \in C_c^\infty(\mathbf{R}^d)$, $0 \leq \varphi \leq 1$, and $\int_{\mathbf{R}^d} \varphi(x) dx = 1$. Let $f_\epsilon = \varphi_\epsilon \star \mu$, then easy computations show that $f_\epsilon \geq 0$, $\int_{\mathbf{R}^d} f_\epsilon(x) dx = \mu(\mathbf{R}^d)$ and that

$$\left| \int_{\mathbf{R}^d} \Delta_h^m \phi(x) f_\epsilon(x) dx \right| = \left| \int_{\mathbf{R}^d} \varphi_\epsilon(x) \left(\int_{\mathbf{R}^d} \Delta_h^m \phi(x-y) \mu(dy) \right) dx \right| \leq K|h|^s \|\phi\|_{C_b^\gamma}.$$

On the other hand, by a discrete integration by parts,

$$(4.2) \quad \int_{\mathbf{R}^d} \Delta_h^m \phi(x) f_\epsilon(x) dx = \int_{\mathbf{R}^d} \Delta_{-h}^m f_\epsilon(x) \phi(x) dx.$$

Set $g_\epsilon = (I - \Delta_d)^{-\beta/2} f_\epsilon$, and $\psi = (I - \Delta_d)^{\beta/2} \phi$, where Δ_d is the d -dimensional Laplace operator and $\beta > \gamma$. We have by [AS61, Theorem 10.1] that $\|g_\epsilon\|_{L^1} \leq c_9 \|f_\epsilon\|_{L^1}$. Moreover, by [Tri83, Theorem 2.5.7, Remark 2.2.2/3]), we know that $C_b^\gamma(\mathbf{R}^d) = B_{\infty,\infty}^\gamma(\mathbf{R}^d)$, and by [Tri83, Theorem 2.3.8] we know that $(I - \Delta_d)^{-\beta/2}$ is a continuous operator from $B_{\infty,\infty}^{\gamma-\beta}(\mathbf{R}^d)$ to $B_{\infty,\infty}^\gamma(\mathbf{R}^d)$. Hence, by (4.2) it follows that

$$\int_{\mathbf{R}^d} \Delta_h^m g_\epsilon(x) \psi(x) dx = \int_{\mathbf{R}^d} \Delta_h^m f_\epsilon(x) \phi(x) dx \leq K|h|^s \|\phi\|_{C_b^\gamma} \leq c_{10} K|h|^s \|\psi\|_{B_{\infty,\infty}^{\gamma-\beta}}$$

Notice that by [Tri83, Theorem 2.11.2], $B_{\infty,\infty}^{\gamma-\beta}(\mathbf{R}^d)$ is the dual of $B_{1,1}^{\beta-\gamma}(\mathbf{R}^d)$, moreover $B_{1,1}^{\beta-\gamma}(\mathbf{R}^d) \hookrightarrow L^1(\mathbf{R}^d)$ by definition, since $\beta > \gamma$, therefore $L^\infty(\mathbf{R}^d) \hookrightarrow B_{\infty,\infty}^{\gamma-\beta}$. By duality, $\|\Delta_h^m g_\epsilon\|_{L^1} \leq c_{10} K|h|^s$, hence $\|g_\epsilon\|_{B_{1,\infty}^s} \leq c_{11}(K + \mu(\mathbf{R}^d))$. Again since $(I - \Delta_d)^{\beta/2}$ maps continuously $B_{\infty,\infty}^s(\mathbf{R}^d)$ into $B_{\infty,\infty}^{s-\beta}(\mathbf{R}^d)$, it finally follows that $\|f_\epsilon\|_{B_{1,\infty}^{s-\beta}} \leq c_{12} \|g_\epsilon\|_{B_{1,\infty}^s}$ for every $\beta > \gamma$.

By Sobolev's embeddings and [Tri83, formula 2.2.2/(18)], we have for every $r < s - \beta$ and $1 \leq p \leq d/(d-r)$ that $B_{1,\infty}^{s-\beta}(\mathbf{R}^d) \hookrightarrow B_{1,1}^r(\mathbf{R}^d) = W^{r,1}(\mathbf{R}^d) \subset L^p(\mathbf{R}^d)$. In particular, $(f_\epsilon)_{\epsilon>0}$ is uniformly integrable in $L^1(\mathbf{R}^d)$, therefore there is f_μ such that $\mu = f_\mu dx$ and $(f_\epsilon)_{\epsilon>0}$ converges weakly in $L^1(\mathbf{R}^d)$ to f_μ . By semi-continuity, (4.1) holds for every $r < s - \gamma$. \square

4.2. The Besov estimate. Let $x \in H$ and consider a solution u of (2.1) that is a limit point of Galerkin approximations. All our estimates will pass to the limit and so it is not restrictive to work on the solution u^N of (2.2) with initial condition $u^N(0) = \pi_N x$.

Given $t > 0$ and $\epsilon \in (0, t)$, let $\chi_{t,\epsilon} = \mathbb{1}_{[0, t-\epsilon]}$ be the indicator function of the interval $[0, t - \epsilon]$, and let u_ϵ^N be the solution of

$$(4.3) \quad du_\epsilon^N + (\pi_N - \pi_F)(\nu A u_\epsilon^N + B(u_\epsilon^N)) dt + \chi_{t,\epsilon} \pi_F B(u_\epsilon^N) dt = \pi_N \mathcal{S} dW,$$

that is $u_\epsilon^N = u^N$ up to time $t - \epsilon$, and $\tilde{u} = \pi_F u_\epsilon^N$ satisfies for $r \in [t - \epsilon, t]$,

$$\tilde{u}(r) = \pi_F u^N(t - \epsilon) + \pi_F \mathcal{S}(W_r - W_{t-\epsilon}).$$

Due to assumption (2.5), $\tilde{u}(r)$ is a d -dimensional Brownian motion (where d is the dimension of F) with spatial covariance matrix $\pi_F \mathcal{S} \mathcal{S}^* \pi_F$.

Proposition 4.2. *Let F be a finite dimensional subspace of $D(A)$ generated by a finite set of eigenvalues of the Stokes operator, and assume (2.6).*

Given $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, there is $c_{13} > 0$ such that if $x \in H$, if N is large enough (that $F \subset H_N$) and u^N is a weak solution of (2.2) with initial condition $\pi_N x$, if $f_N(\cdot; x)$ is the density with respect to the Lebesgue measure on F of the random variable $\pi_F u^N(\cdot)$, then

$$\|f_N(t) - f_N(s)\|_{B_{1,\infty}^\alpha} \leq c_{13} |t - s|^{\frac{\beta}{2}},$$

for every $s, t > 0$, where

$$c_{13} \approx (1 + s \vee t)^{\frac{1-\beta}{2}} (1 + \|x\|_H^2)^3 ([f_N(t)]_{B_{1,\infty}^{1-\delta}} + [f_N(s)]_{B_{1,\infty}^{1-\delta}}),$$

and $\delta < 1 - (\alpha + \beta)$.

The following lemma summarizes the result of [DR14], adding the explicit dependence of the Besov norm of the density in terms of time, which is needed for the evaluation of the inequality in the previous proposition.

Lemma 4.3. *Let F be a finite dimensional subspace of $D(A)$ generated by a finite set of eigenvalues of the Stokes operator, and assume (2.5). For every $t > 0$ and $x \in H$, the projection $\pi_F u(t)$ has a density $f_F(t)$ with respect to the Lebesgue measure on F , where u is any solution of (2.1), with initial condition x , which is a limit point of the spectral Galerkin approximations.*

Moreover, for every $\alpha \in (0, 1)$, $f_F(t) \in B_{1,\infty}^\alpha(\mathbb{R}^d)$ and for every (small) $\epsilon > 0$, there exists $c_{14} = c_{14}(\alpha, \epsilon) > 0$ such that

$$\|f_F(t)\|_{B_{1,\infty}^\alpha} \leq \frac{c_{14}}{(1 \wedge t)^{\alpha+\epsilon}} (1 + \|x\|_H^2)^{\alpha+\epsilon}.$$

Proof. Given a finite dimensional space F as in the statement, fix $t > 0$, and let $\gamma \in (0, 1)$, $\phi \in C_b^\gamma$, and $h \in F$, with $|h| \leq 1$. For $m \geq 1$, consider two cases. If $|h|^{2n/(2\gamma+n)} < t$, then we use the same estimate in [DR14] to get

$$|\mathbb{E}[\Delta_h^m \phi(\pi_F u(t))]| \leq c_{15} (1 + \|x\|_H^2)^\gamma \|\phi\|_{C_b^\gamma} |h|^{\frac{2n\gamma}{2\gamma+n}}.$$

If on the other hand $t \leq |h|^{2n/(2\gamma+n)}$, we introduce the process u_ϵ as above, but with $\epsilon = t$. As in [DR14],

$$\mathbb{E}[\Delta_h^m \phi(\pi_F u(t))] = \mathbb{E}[\Delta_h^m \phi(\pi_F u_\epsilon(t))] + \mathbb{E}[\Delta_h^m \phi(\pi_F u(t)) - \Delta_h^m \phi(\pi_F u_\epsilon(t))]$$

and

$$|\mathbb{E}[\Delta_h^m \phi(\pi_F u(t)) - \Delta_h^m \phi(\pi_F u_\epsilon(t))]| \leq c_{16} (1 + \|x\|_H^2)^\gamma \|\phi\|_{C_b^\gamma} t^\gamma.$$

For the probabilistic error we use the fact that $u_\epsilon(t)$ is Gaussian, hence

$$|\mathbb{E}[\Delta_h^m \phi(\pi_F u_\epsilon(t))]| \leq c_{17} \|\phi\|_\infty \left(\frac{|h|}{\sqrt{t}} \right)^{\frac{2n\gamma}{2\gamma+n}}$$

In conclusion, from both cases we finally have

$$|\mathbb{E}[\Delta_h^m \phi(\pi_F u(t))]| \leq c_{18} (1 + \|x\|_H^2)^\gamma \|\phi\|_{C_b^\gamma} |h|^{\frac{2n\gamma}{2\gamma+n}} (1 \wedge t)^{-\frac{n\gamma}{2\gamma+n}}.$$

Given α , suitable choices of n and γ yield the final result. \square

Lemma 4.4. *Let $\beta_r = \pi_F \mathcal{S} W_r$, $r \geq 0$. There is $c_{19} > 0$ such that*

$$|\mathbb{E}[\Delta_h^n \phi(a + \beta_r) - \Delta_h^n \phi(a + \beta_s)]| \leq \frac{c_{19}}{r \vee s} \|\phi\|_\infty \left(\frac{|h|}{\sqrt{r \wedge s}} \right)^n |r - s|,$$

for every $a \in F$, $n \geq 1$, $\phi \in C_c^\infty(F)$, $h \in F$ with $|h|_F \leq 1$, and $r, s \geq 0$.

Proof. By assumption (2.5), β is a d -dimensional Brownian motion with covariance matrix $\pi_F \mathcal{S} \mathcal{S}^* \pi_F$. If Q is a $d \times d$ matrix such that $\pi_F \mathcal{S} \mathcal{S}^* \pi_F = QQ^*$, then $\beta_r = QB_r$, where B_r is a standard d -dimensional Brownian motion. The position $\psi(x) = \phi(a + Qx)$ reduces the statement to the same for a standard Brownian motion. The latter is a straightforward estimate. \square

In the rest of the section we will drop, for simplicity and to make the notation less cumbersome, the index N . It is granted though that we work with solutions of the Galerkin system (2.2).

Lemma 4.5. *Assume (2.4) and let v be the process introduced in Section 3.1. Given $\gamma \in (0, 1)$, there exists $c_{20} > 0$ such that for every $0 < s \leq t$ and every bounded measurable $\psi : F \rightarrow \mathbb{R}$,*

$$(4.4) \quad |\mathbb{E}[\psi(\pi_F u(t)) - \psi(\pi_F u(s))] - \mathbb{E}[\psi(\pi_F v(t)) - \psi(\pi_F v(s))]| \leq \\ \leq c_{20} (1 + \|u(0)\|_H^2)^2 \log(2 + \|u(0)\|_H^2) \sqrt{t} (-\log(\frac{1}{2} \wedge t)) [\psi]_{C_b^\gamma} (t - s)^{\frac{\gamma}{2}}.$$

Proof. We work in the framework introduced in Section 3.1. Let us denote, for brevity, the left-hand side of (4.4) by $\boxed{\text{num}}$. We have that

$$\boxed{\text{num}} = \mathbb{E}[G_t^n (\psi(\pi_F v^n(t)) - \psi(\pi_F v^n(s)))] - \mathbb{E}[\psi(\pi_F v(t)) - \psi(\pi_F v(s))]$$

First we notice that we can replace v^n by v in the above formula, up to an error that converges to 0 as $n \rightarrow \infty$. Indeed, by Lemma 3.4, for every $\delta > 0$,

$$|\mathbb{E}[G_t^n \psi(\pi_F v^n(t))] - \mathbb{E}[G_t^n \psi(\pi_F v(t))]| \leq \\ \leq \delta \|\psi\|_\infty (c_{22} \sqrt{t} (1 + \|x\|_H^2)^2 + e^{\frac{2}{\delta}} \mathbb{P}[\tau_n(v) < t]),$$

and likewise at time s , where $u(0) = x$. After replacing v^n by v we will obtain an estimate that is uniform in n . By taking first the limit as $n \rightarrow \infty$ and then as $\delta \downarrow 0$, the lemma will be proved.

After this preliminary observation, we see that

$$\boxed{\text{num}} \approx \mathbb{E}[(G_t^n - 1)(\psi(\beta_t) - \psi(\beta_s))],$$

where, as in Lemma 3.5, $\beta_t = \pi_F v(t)$ is a d -dimensional Brownian motion. By (3.1), for given $a, b > 0$ that will be given later,

$$\boxed{\text{num}} \lesssim ab \mathbb{E}\left[\frac{|G_t^n - 1|}{a} \log \frac{|G_t^n - 1|}{a}\right] + ab \mathbb{E}\left[\exp\left(\frac{|\psi(\beta_t) - \psi(\beta_s)|}{b}\right)\right] = ab \boxed{1} + ab \boxed{2}.$$

Notice that

$$|\psi(\beta_t) - \psi(\beta_s)| \leq [\psi]_{C_b^\gamma} |\beta_t - \beta_s|^\gamma = [\psi]_{C_b^\gamma} |t - s|^{\frac{\gamma}{2}} |Z|^\gamma,$$

where Z is a Gaussian random variable whose distribution does not depend on s, t . If we choose $b = [\psi]_{C_b^\gamma} |t - s|^{\frac{\gamma}{2}}$, then $\boxed{2} \leq \mathbb{E}[\exp(|Z|^\gamma)] \leq c_{23}$.

For the first term $\boxed{1}$ we see that

$$\boxed{1} = \frac{1}{a} \mathbb{E}[|G_t^n - 1| \log |G_t^n - 1|] - \frac{\log a}{a} \mathbb{E}[|G_t^n - 1|].$$

The same argument of Lemma 3.3 (here $G_t^n - 1 = G_t^n - G_0^n$) yields

$$\mathbb{E}[|G_t^n - 1|] \leq c_{24}(1 + \|x\|_H^2)^2 \sqrt{t}$$

Moreover, by Lemma 3.2,

$$\begin{aligned} \mathbb{E}[|G_t^n - 1| \log |G_t^n - 1|] &= \\ &= \mathbb{E}[(G_t^n - 1) \log(G_t^n - 1) \mathbb{1}_{\{G_t^n \geq 2\}}] + \underbrace{\mathbb{E}[|G_t^n - 1| \log |G_t^n - 1| \mathbb{1}_{\{G_t^n \leq 2\}}]}_{\leq 0} \leq \\ &\leq \mathbb{E}[G_t^n \log G_t^n \mathbb{1}_{\{G_t^n \geq 2\}}] \leq \mathbb{E}[G_t^n |\log G_t^n|] \leq c_{26} \sqrt{t} (1 + \|x\|_H^2)^2. \end{aligned}$$

In conclusion

$$\boxed{1} \leq c_{27} \frac{\sqrt{t}}{a} (1 + \|x\|_H^2)^2 (1 + |\log a|).$$

The choice

$$a \approx (1 + \|x\|_H^2)^2 \log(2 + \|x\|_H^2) \sqrt{t} (-\log(\frac{1}{2} \wedge t)),$$

yields $\boxed{1} \leq c_{28}$. □

Proof of Proposition 4.2. Denote by f the density of $\pi_F u$. Let $s \leq t$, $\phi \in C_c^\infty(\mathbf{R}^d)$, $h \in F$ with $|h|_F \leq 1$, and fix the parameters $\gamma, \delta \in (0, 1)$, $\epsilon > 0$, $n \geq 3$ that will be chosen along the proof.

Assume that $t - s \leq |h|^2$, then by a discrete integration by parts,

$$\begin{aligned}
 \int_F \phi \Delta_h^n(f(t) - f(s)) \, dx &= \int_F (f(t) - f(s)) \Delta_{-h}^n \phi \, dx \\
 &= \mathbb{E}[\Delta_{-h}^n \phi(\pi_F u(t)) - \Delta_{-h}^n \phi(\pi_F u(s))] \\
 &= \underbrace{\mathbb{E}[\Delta_{-h}^n \phi(\pi_F u(t)) - \Delta_{-h}^n \phi(\pi_F u_\epsilon(t))]}_{\text{num}_t} \\
 &\quad + \underbrace{\mathbb{E}[\Delta_{-h}^n \phi(\pi_F u_\epsilon(t)) - \Delta_{-h}^n \phi(\pi_F u_\epsilon(s))]}_{\text{prob}} \\
 &\quad + \underbrace{\mathbb{E}[\Delta_{-h}^n \phi(\pi_F u_\epsilon(s)) - \Delta_{-h}^n \phi(\pi_F u(s))]}_{\text{num}_s},
 \end{aligned}$$

where u_ϵ has been defined in (4.3).

To estimate prob , we first point out that we will choose ϵ so that $t - s \leq \frac{\epsilon}{2}$. Notice that

$$\text{prob} = \mathbb{E}[\mathbb{E}[\Delta_{-h}^n \phi(\pi_F u_\epsilon(t)) - \Delta_{-h}^n \phi(\pi_F u_\epsilon(s)) \mid \mathcal{F}_{t-\epsilon}]],$$

and that, given $\mathcal{F}_{t-\epsilon}$, $\pi_F u_{N,\epsilon}(r)$ has the same law of $\pi_F u(t - \epsilon) + \beta_{r-t+\epsilon}$, where β is the process of Lemma 4.4. Hence, by Lemma 4.4, and since $t - s \leq \frac{\epsilon}{2}$,

$$\begin{aligned}
 (4.5) \quad \text{prob} &= \mathbb{E}[\mathbb{E}[\Delta_{-h}^n \phi(\pi_F u(t - \epsilon) + \beta_\epsilon) - \Delta_{-h}^n \phi(\pi_F u(t - \epsilon) + \beta_{s-t+\epsilon}) \mid \mathcal{F}_{t-\epsilon}]] \\
 &\leq \frac{c_{29}}{\epsilon^{1+\frac{n}{2}}} \|\phi\|_\infty |h|^n |t - s|.
 \end{aligned}$$

Let $\text{num} = \text{num}_s + \text{num}_t$, then by conditioning

$$\begin{aligned}
 \text{num} &= \mathbb{E}[\mathbb{E}[\Delta_{-h}^n \phi(\pi_F u(t)) - \Delta_{-h}^n \phi(\pi_F u(s)) \mid \mathcal{F}_{t-\epsilon}]] + \\
 &\quad - \mathbb{E}[\mathbb{E}[\Delta_{-h}^n \phi(\pi_F u_\epsilon(t)) - \Delta_{-h}^n \phi(\pi_F u_\epsilon(s)) \mid \mathcal{F}_{t-\epsilon}]].
 \end{aligned}$$

We use the Markov property and Lemma 4.5 with times $s - t + \epsilon$ and ϵ , and $\psi = \Delta_{-h}^n \phi$ to get

$$\begin{aligned}
 (4.6) \quad \text{num} &\leq c_{20} \mathbb{E}[(1 + \|u(t - \epsilon)\|_H^2)^2 \log(2 + \|u(t - \epsilon)\|_H^2)] \sqrt{\epsilon} (-\log \epsilon) [\phi]_{C_b^\gamma}(t - s)^{\frac{\gamma}{2}} \\
 &\leq c_{30} (1 + \|x\|_H^2)^3 \sqrt{\epsilon} (-\log \epsilon) [\phi]_{C_b^\gamma}(t - s)^{\frac{\gamma}{2}}.
 \end{aligned}$$

In conclusion (4.5) and (4.6) yield

$$\left| \int_F \phi \Delta_h^n(f(t) - f(s)) \, dx \right| \leq c_{31} (1 + \|x\|_H^2)^3 \|\phi\|_{C_b^\gamma}(t - s)^{\frac{\gamma}{2}} \left(\epsilon^{\frac{1}{2}(1-\delta)} + \frac{|h|^n}{\epsilon^{1+\frac{n}{2}}} \right),$$

where $\delta \in (0, 1)$ has been introduced to get rid of the log correction and simplify computations. By optimizing in ϵ we choose $\epsilon^{\frac{1}{2}(n+3-\delta)} \sim |h|^n$, that is $\epsilon \sim |h|^{\frac{2n}{n+3-\delta}}$

(the exponent of $|h|$ is smaller than 2, hence $(t - s) \lesssim \epsilon$ and $(t - s)$ can be made smaller than $\frac{\epsilon}{2}$ by a suitable constant). We finally have

$$(4.7) \quad \left| \int_{\mathbb{F}} \phi \Delta_h^n (f(t) - f(s)) \, dx \right| \leq c_{32} (1 + \|x\|_H^2)^3 \|\phi\|_{C_b^\gamma} (t - s)^{\frac{\gamma}{2}} |h|^{E_n},$$

with $E_n = \frac{n(1-\delta)}{n+3-\delta} \uparrow 1 - \delta$.

If on the other hand $t - s \geq |h|^2$, by integrating by parts once in the discrete variable,

$$\begin{aligned} \left| \int_{\mathbb{F}} \phi \Delta_h^n (f(t) - f(s)) \, dx \right| &= \left| \int_{\mathbb{F}} (\Delta_{-h} \phi) \Delta_h^{n-1} (f(t) - f(s)) \, dx \right| \\ &\leq \|\Delta_h^{n-1} (f(t) - f(s))\|_{L^1(\mathbb{F})} \|\Delta_{-h} \phi\|_{\infty} \\ &\leq [f(t) - f(s)]_{B_{1,\infty}^{1-\delta}} [\phi]_{C_b^\gamma} |h|^{1-\delta+\gamma} \\ &\leq ([f(t)]_{B_{1,\infty}^{1-\delta}} + [f(s)]_{B_{1,\infty}^{1-\delta}}) \|\phi\|_{C_b^\gamma} |h|^{1-\delta+\gamma}. \end{aligned}$$

Since $|h|^2 \leq (t - s)$, $|h| \leq 1$, and $E_n \leq 1 - \delta$, $|h|^{1-\delta+\gamma} \leq (t - s)^{\frac{\gamma}{2}} |h|^{E_n}$, and we finally get

$$(4.8) \quad \left| \int_{\mathbb{F}} \phi \Delta_h^n (f(t) - f(s)) \, dx \right| \leq ([f(t)]_{B_{1,\infty}^{1-\delta}} + [f(s)]_{B_{1,\infty}^{1-\delta}}) \|\phi\|_{C_b^\gamma} (t - s)^{\frac{\gamma}{2}} |h|^{E_n}.$$

We have all the ingredients to conclude the proof. Let $\beta \in (0, 1)$ and $\alpha < 1 - \beta$, and choose $\gamma = \beta$. Choose δ small enough and n large enough that $E_n \geq \alpha + \beta$. Then Proposition 3.1 and the same arguments of Lemma 4.1 yield that

$$\|f(t) - f(s)\|_{B_{1,\infty}^\alpha} \leq c_{13} (t - s)^{\frac{\beta}{2}},$$

where c_{13} is the sum of the contribution from Proposition 3.1 and the maximum between the contributions from (4.7) and (4.8). \square

Remark 4.6. A worse estimate can be obtained if one want to avoid Girsanov's transformation and assumption (2.4), and rely only on assumption (2.5) (at least when giving an estimate of the Besov seminorm). Indeed, instead of using Lemma 4.5, we estimate the $\boxed{\text{num}}$ terms in two different ways, to take into account both the control by $|t - s|$ and by ϵ . On the one hand, to estimate $\boxed{\text{num}_s}$ and $\boxed{\text{num}_t}$, notice that if $r \in [t - \epsilon, t]$,

$$(4.9) \quad \boxed{\text{num}_r} \leq 2^n [\phi]_{C_b^\gamma} \mathbb{E}[\|\pi_F u(r) - \pi_F u_\epsilon(r)\|_H^\gamma] \leq c_{33} (1 + \|x\|_H^2)^\gamma \epsilon^\gamma [\phi]_{C_b^\gamma},$$

since

$$\pi_F u(r) - \pi_F u_\epsilon(r) = - \int_{t-\epsilon}^r (\nu \pi_F A u(\rho) + \pi_F B(u(\rho))) \, d\rho,$$

hence, by (2.3),

$$\mathbb{E}[\|\pi_F u(r) - \pi_F u_\epsilon(r)\|_H] \leq c_{34} \int_{t-\epsilon}^r (1 + \mathbb{E}[\|u(\rho)\|_H^2] \, d\rho) \leq c_{35} \epsilon (1 + \|x\|_H^2).$$

On the other hand,

$$(4.10) \quad \boxed{\text{num}_s} + \boxed{\text{num}_t} \leq c_{36}[\Phi]_{C_b^\gamma} (1 + t^{\frac{\gamma}{2}})(1 + \|x\|_H^2)^\gamma (t - s)^{\frac{\gamma}{2}},$$

since

$$\pi_F(u(t) - u(s)) = -\pi_F \int_s^t (\nu A u(r) + B(u(r))) dr + \pi_F \mathcal{S}(W_t - W_s),$$

hence by (2.3) and standard estimates on the Wiener process,

$$\mathbb{E}[\|\pi_F u(t) - \pi_F u(s)\|_H] \leq c_{37}(1 + \sqrt{t})(1 + \|x\|_H^2)\sqrt{t - s},$$

(and likewise but simpler for the increment of u_ϵ).

In conclusion, using (4.5), (4.9), and (4.10), for every $\lambda \in (0, 1)$,

$$\begin{aligned} \left| \int_F \Phi \Delta_h^n(f(t) - f(s)) dx \right| &\leq c_{38} \|\Phi\|_{C_b^\gamma} (1 + \|x\|_H^2)^\gamma (1 + t)^{\frac{1}{2}\gamma(1-\lambda)} \\ &\quad \cdot \left(\epsilon^{\lambda\gamma} (t - s)^{\frac{1}{2}\gamma(1-\lambda)} + \frac{|h|^n}{\epsilon^{\frac{n+2}{2}}} (t - s) \right). \end{aligned}$$

Optimize in ϵ and choose $\epsilon \sim |h|^{\frac{2n}{n+\gamma(1+\lambda)}}$ (the exponent is smaller than 2, hence $(t - s) \lesssim \epsilon$ and can be made smaller than $\frac{\epsilon}{2}$ by a suitable constant) to get

$$(4.11) \quad \begin{aligned} \left| \int_F \Phi \Delta_h^n(f(t) - f(s)) dx \right| &\leq \\ &\leq c_{40} \|\Phi\|_{C_b^\gamma} (1 + \|x\|_H^2)^\gamma (1 + t)^{\frac{1}{2}\gamma(1-\lambda)} (t - s)^{\frac{1}{2}\gamma(1-\lambda)} |h|^{\frac{2n\gamma\lambda}{n+\gamma(1+\lambda)}}. \end{aligned}$$

The case $t - s \geq |h|^2$ yields, as in the previous proof,

$$(4.12) \quad \begin{aligned} \left| \int_F \Phi \Delta_h^n(f(t) - f(s)) dx \right| &\leq \\ &\leq ([f(t)]_{B_{1,\infty}^\gamma} + [f(s)]_{B_{1,\infty}^\gamma}) \|\Phi\|_{C_b^\gamma} (t - s)^{\frac{1}{2}\gamma(1-\lambda)} |h|^{\frac{2n\gamma\lambda}{n+\gamma(1+\lambda)}}. \end{aligned}$$

The estimate in L^1 would be as follows: given $a, b > 0$ with $a + 2b < 1$, choose $\gamma \in (a + 2b, 1)$ and $\lambda = 1 - b/\gamma$, so that $a < (2\lambda - 1)\gamma$, hence there is n large enough such that

$$\frac{2n\lambda\gamma}{n + \gamma(1 + \lambda)} - \gamma \leq a,$$

and by using (4.11) and (4.12), the same arguments of Lemma 4.1 and Proposition 3.1 yield that

$$\|f(t) - f(s)\|_{B_{1,\infty}^a} \leq c_{41}(s, t, a, b)(t - s)^{\frac{1}{2}b}.$$

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